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LATERAL BUCKLING OF I BEAMS UNDER THRUST AND UNEQUAL END MOMENTS

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LATERAL BUCKLING OF I BEAMS UNDER THRUST AND UNEQUAL END MOMENTS

Mario G. Salvadori, A.M. ASCE(*)

Introduction

A slender beam, acting as a member of a framed structure, may buckle in a variety of ways, among which the following are of practical interest:

1) In bending under axial thrusts P (Euler buckling);
2) In bending and torsion (lateral buckling) under the action of end-moments due to the continuity of the structure;
3) In bending and torsion under a combination of axial thrusts and end-moments, in which case the moments may be due either to the continuity of the structure or to eccentricities of the thrusts.

The solution of the lateral instability problem in case 3) involves the determination of the critical combination of thrusts and moments and hence of so-called interaction curves from which the critical value of the moments may be derived for a given thrust or the critical value of the thrust for given end-moments.

Interaction curves for thrusts and equal end-moments have been given by Timoshenko[1] for beams of rectangular cross-section and by Johnston[2] for I beams. In both cases the interaction curves take the simple form

$$\left(\frac{P}{P_{cr}}\right)^2 + \left(\frac{M}{M_{cr}}\right)^2 = 1 \qquad (1)$$

where P_{cr} is Euler's load and M_{cr} the critical value of the end-moment without thrust.

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The critical value of unequal end-moments was computed by Salvadori[3] for beams of rectangular cross-section and by Thomas[4,5] for I beams. Interaction curves for beams of rectangular cross-section under thrust and unequal end-moments were given by DiMaggio[5,6].

In the present paper interaction curves are derived for I beams under thrust and unequal end-moments by the Rayleigh-Ritz method.

Lateral Buckling Energy of I Beams.

The total energy of an I beam buckled laterally under the action of normal loads and longitudinal thrusts is given by[1, 7, 8]

$$V = \frac{B}{2}\int_o^l (u'')^2 dz + \frac{Dh^2}{4}\int_o^l (\beta'')^2 dz + \frac{H}{2}\int_o^l (\beta')^2 dz + \int_o^l M\beta u'' dz - \frac{P}{2}\int_o^l (u')^2 dz \quad (2)$$

where

B = minimum flexural rigidity of beam

u = lateral deflection of beam in the plane of minimum rigidity

l = beam length

D = flexural rigidity of one flange in its own plane

h = beam depth

β = angular rotation of section of abscissa z

H = C - (PI_p/A) = equivalent torsional rigidity of beam

C = torsional rigidity of beam

P = thrust, uniformly distributed on beam cross-section

I_p = polar moment of inertia of beam cross-section about its centroid

A = cross-sectional area of beam

M = bending moment in the plane of maximum flexural rigidity at section of abscissa z

z = abscissa measured along the beam axis with origin at one of its ends.

This expression was proved by H.H. Bleich[9] to be rigorously correct under the assumption that the thrust P be evenly distributed over the end sections of the beam, but may be used with sufficient accuracy even if P is otherwise distributed, e.g., is concentrated at the centroid of the section. The equivalent torsional rigidity H takes care of the weakening of the torsional rigidity C due to the tendency of a compressed beam to buckle in pure torsion when $P = AC/I_p$. In most practical cases H is equal to C. The term of Eq. (2) containing the bending moment M(z) in the plane of the maximum flexural rigidity represents the work of the lateral loads applied to the beam and its use is limited to beams simply supported or built-in on either end, or to cantilevered beams.

The two differential equations in u and β governing the buckling of the beam may be easily derived from Eq. (2) by calculus of variations together with the natural boundary conditions of the problem. But since the differential equations in u and β are not readily integrable in finite terms, it is more practical to obtain interaction curves by means of the Rayleigh-Ritz method.

Lateral Buckling for Simply Supported I Beams.

Considering in particular "simply supported" beams, whose end sections are free to rotate around their principal axes but are prevented from twisting around the longitudinal beam axis, the boundary contitions for u and β become:

$$u(0) = u''(0) = u(l) = u''(l) = 0 \qquad (3a)$$

$$\beta(0) = \beta''(0) = \beta(l) = \beta''(l) = 0 \qquad (3b)$$

and u and β may be expanded into sine series:

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi z/l \quad ; \qquad \beta = \sum_{n=1}^{\infty} b_n \sin n\pi z/l \qquad (4)$$

Indicating by M_2 the value of M(z) at the end $z = l$, by M_1 the value of M(z) at the end $z = 0$, and setting:

$$r = M_1/M_2 \qquad (5)$$

the bending moment M(z) becomes in this case:

$$M(z) = M_2 [\, r + (1 - r)z/l \,]. \qquad (6)$$

Taking into account an increasing number of terms of the series (4), substitution of these values for u and β in Eq. (2) gives a quadratic form in the a_n and b_n. The partial derivatives of this form with respect to the a_n and b_n equated to zero give a set of linear simultaneous equations in the a_n and b_n, whose determinant set equal to zero determines couples of critical values of P and M_2 as functions of the parameters r of Eq. (5) and

$$\frac{l^2}{a^2} = \frac{2\, l^2 C}{h^2 D}. \qquad (7)$$

The tables and graphs of this paper give the results of the computations in terms of the moment coefficient

$$K = \frac{M_2\, l}{\sqrt{BH}} \qquad (8)$$

and of the nondimensional thrust

$$p \equiv \frac{P}{P_e} = \frac{P \ell^2}{\pi^2 B} \tag{9}$$

for the following values of the parameters:

$$r = 1, \frac{1}{2}, 0, -\frac{1}{2}, -1 \tag{10}$$

$$\frac{\ell^2}{a^2} = 0.1, 1, 2, 6, 10, 16, 24, 32, 40, 100, \infty . \tag{11}$$

The cases $r = 1$, corresponding to equal end-moments, and the cases $\ell^2/a^2 = \infty$, corresponding to beams of rectangular cross-section, were taken from the papers by Timoshenko[1], Johnston[2] and DiMaggio[5,6]. The cases $p = 0$, corresponding to lack of thrust, were taken from the papers by Salvadori[3] and Thomas[4,5].

The values of K for

$$p = 0, 0.2, 0.4, 0.6, 0.8, 1.0 \tag{12}$$

were obtained by solution of the determinantal equation taking into account 1 term of each series (4) for $r = 1$, 3 terms for $r = \frac{1}{2}$, $r = 0$ and $r = -\frac{1}{2}$ (although 2 terms would have been sufficient for $r = \frac{1}{2}$), and 4 terms for $r = -1$. The successive approximations indicate that the results are correct within the number of figures given in the tables. Whatever the number of terms of the series taken into account, it is found that the determinantal equation for $r = -1$ splits into two separate equations, whose common root occurs for values of p varying between 0.949 and 0.759 as ℓ^2/a^2 varies from 0.1 to ∞. These results indicated a change of equilibrium configuration for these values of p and required the solution of the determinantal equation for $p = 0.95$ when $r = -1$, in order to obtain more accurate K-graphs.

The K versus p graphs, one for each value (11) of ℓ^2/a^2, contain 5 curves corresponding to the 5 values (10) of r.

The Tables, one for each value (11) of ℓ^2/a^2, give K versus p and r: p is given the values (12), while r varies in steps of 0.1 from + 1.0 to - 1.0. The values of K corresponding to the intermediate values of r, not included among the 5 values (10), were obtained by lagrangian interpolation by means of the values corresponding to the values (10) of r. The interpolated values have a maximum error of between 1% and 2%.

The graphs and the tables can be used directly when P is given and M_2 is to be determined. To solve for P for a given M_2, one computes first an approximate value K' of K by substituting C for H in Eq. (8). With this value of K one determines from the tables or graphs the corresponding value of p and hence of $P = p\pi^2 B/\ell^2$, which allows the evaluation of $H = C - (PI_p/A)$ and hence of a more accurate value of K. A new value of p is then derived from the tables or graphs, but it is seldom that this procedure must be iterated.

When the end-moments are due to eccentricities e_1 and e_2 of the thrusts at z = 0 and z = ℓ, respectively, these may be determined by noticing that

$$e_2 = \frac{M_2}{P} \qquad e_1 = re_2 \tag{13}$$

Lower Bounds of Interaction Curves for Continuous Beams.

The interaction curves of the present paper may be conveniently used to determine lower, i.e., safe bounds for thrusts and end-moments of continuous beams, as shown by the following simple examples.

a) In Fig. 12, an I beam with 2 identical spans of $l^2/a^2 = 6$, built-in at the left support and simply supported at the right, is loaded by an **eccentric thrust** P. We wish to determine a lower bound for the **eccentricity** e, when $P/P_e = 0.6$, where P_e is Euler's load for each span **considered as simply supported. Letting $P_e = M_{(2)}$, 2 three** moment equations give:

$$2\,M_{(0)} + M_{(1)} = 0$$

$$M_{(0)} + 4\,M_{(1)} + M_{(2)} = 0$$

From which

$$M_{(0)} = \frac{1}{7} M_{(2)} \quad , \quad M_{(1)} = -\frac{2}{7} M_{(2)} \qquad (14\ a,b)$$

Hence the ratio r of the left span equals $r' = -\frac{1}{2}$ while the ratio r of the right span is $r'' = -\frac{2}{7}$.

If we now consider each of the two spans as simply supported but loaded by the bending moment diagram of the continuous beam and evaluate the critical values of their **end-moments, these will be upper and lower bounds of the corresponding moment for the continuous beam, due to the lack of continuity in the x,z-plane.**

For the left span, with $p = 0.6$ and $r = -\frac{1}{2}$, Table 4 gives:

$$K_1 = \frac{M_2\, l}{\sqrt{BH}} = 9.91$$

or

$$M_{(1)} = -9.91\,\frac{\sqrt{BH}}{l} \qquad (15)$$

from which, by Eq. (14 b), the critical value of $M_{(2)}$ becomes:

$$M'_{(2)} = -\frac{7}{2} M_{(1)} = 34.63\,\frac{\sqrt{BH}}{l} \qquad (16)$$

For the right span with p = 0.6 and $r = -\frac{2}{7} \doteq -0.30$, Table 4 gives

$$K_2 = \frac{M_2 l}{\sqrt{BH}} = 8.30$$

or

$$M''_{(2)} = 8.30 \frac{\sqrt{BH}}{l} \qquad (17)$$

Since $M'_{(2)} > M''_{(2)}$ the left span stiffens the right span and the right span weakens the left. Hence the critical value of $M_{(2)}$ is certainly greater than $8.30 \sqrt{BH}/l$ and less than $34.68 \sqrt{BH}/l$. The lower bound is usually much nearer to the actual $M_{(2)\,cr}$ than the upper bound. Hence

$$e = \frac{8.30 \sqrt{BH}/l}{0.6\pi^2 B/l^2} = 1.40 \sqrt{H/B}\, l \qquad (18)$$

is a safe approximate bound for the eccentricity of a load $P = 0.6 P_e$.

b) In Fig. 13 a beam continuous on 3 supports is loaded by end-moments $M_{(0)}$ and $M_{(2)} = 3 M_{(0)}$; we wish to determine a lower bound for the critical thrust P, if

$$\frac{l^2}{a^2} = 10 \; ; \quad M_{(0)} = 3 \frac{\sqrt{BH}}{l} \; ; \quad M_{(2)} = 9 \frac{\sqrt{BH}}{l} \qquad (19)$$

A three-moment equation gives in this case:

$$M_{(0)} = -M_{(1)} = \frac{1}{3} M_{(2)} \qquad (20)$$

so that the value of r for the left and right span becomes, respectively

$$r' = -1 \; ; \qquad r'' = -\frac{1}{3} \qquad (21)$$

The curve $r = -1$ of Fig. 5 ($l^2/a^2 = 10$) gives for $K = 3$ a value of $p \doteq 1$ or a thrust $P = \pi^2 B/l^2$.

The same figure or Table 5 gives, for K = 9 and r = - 0.33 (interpolating linearly first between r = - 0.3 and r = - 0.4 for p = 0.2 and p = 0.4, and then between p = 0.2 and p = 0.4 for r = - 0.33) p = 0.32 or a thrust $P = 0.32\ \pi^2\ B/l^2$. Here again the left span stiffens the right span and

$$P_{cr} > 3.16\ B/l^2 \qquad (22)$$

(It was assumed throughout this second example that H is practically equal to C).

c) The frame of Fig. 14a is loaded by a horizontal force P. We wish to determine a lower bound for the critical value of this force, assuming that the frame is prevented from buckling laterally as a whole.

The bending moment diagram of Fig. 14b shows that the ratio r has the value r" = - 1 for the horizontal beam and r' = r" = 0 for the columns.

With an $(\frac{l}{a})^2$ equal to 2 for the horizontal beam and assuming this beam simply supported, the critical value of P may be found as follows:

Since $M_2" = \frac{Pl}{2}$, the parameter p becomes, in this case,

$$p = p" = \frac{P}{P_e"} = \frac{2}{l} M_2" \frac{L^2}{\pi^2 B_2} = \frac{2}{l} \frac{K" \sqrt{B_2 H_2}}{L} \frac{L^2}{\pi^2 B_2} =$$

$$= K" \frac{2}{\pi^2} \sqrt{\frac{H_2}{B_2}}\ \frac{L}{l} \qquad (23)$$

Letting

$$\alpha" = \frac{\pi^2}{2} \sqrt{\frac{B_2}{H_2}}\ \frac{l}{L} \quad , \qquad (24)$$

the ratio of K" to p" becomes

$$K"/p" = \alpha" \qquad (25)$$

Hence, from Fig. 3, corresponding to $(\frac{l}{a})^2 = 2$, we must find the point on the curve r = - 1 for which K/p = α'', i.e., the intersection of the curve r = - 1 with a straight line through the origin of slope α''. With $\alpha'' = 15$, for example, we obtain K" = 14.50 and p" = 0.967 and hency by (23):

$$P'' = \begin{cases} \dfrac{2K''\sqrt{B_2H_2}}{lL} = \dfrac{29.00\sqrt{B_2H_2}}{lL} \\ p''\,P_e'' = 0.967\,\dfrac{\pi^2 B_2}{L^2} \end{cases} \qquad (26)$$

Similarly, assuming an $(\frac{l}{a})^2 = 6$ for the columns and considering the right column, which is under compression, we have, with $M_2''' = \frac{Pl}{2}$,

$$p = p''' = \frac{P}{P_e'''}\,\frac{l}{L} = \frac{2}{L}\,M_2'''\,\frac{l^2}{\pi^2 B_1} = \frac{2}{L}\,\frac{K'''\sqrt{B_1H_1}}{l}\,\frac{l^2}{\pi^2 B_1} =$$

$$= K'''\,\frac{2}{\pi^2}\sqrt{\frac{H_1}{B_1}}\,\frac{l}{L} = K'''/\alpha''' \qquad (27)$$

With $\alpha''' = 10$, for example, the curve r = 0 of Fig. 4 [$(\frac{l}{a})^2 = 6$] gives K''' = 6.07 and p''' = 0.61. Hence:

$$p''' = \begin{cases} \dfrac{2K'''\sqrt{B_1H_1}}{lL} = \dfrac{12.14\sqrt{B_1H_1}}{lL} \\ p'''\,\dfrac{L}{l}\,P_e''' = 0.61\,\dfrac{L}{l}\,\dfrac{\pi^2 B_1}{l^2} \end{cases} \qquad (28)$$

The smaller of the two loads P" and P''' is a lower bound for the critical value of P.

Evaluation of Tables

The tables and graphs of this paper were evaluated numerically by the staff of the Istituto Nazionale di Calcolo, the mathematical laboratory of the Italian National Research Council directed by Professor Mauro Picone. The calculations were supervised by Drs. Caligo and Roma and were performed by Messrs. Bellino and Gullotta. I am greatly indebted to the Istituto di Calcolo for this careful numerical evaluation, which was partially supported by a grant of the Column Research Council of the Engineering Foundation, of which Mr. S. Hardesty is Chairman.

Calculations of interaction curves for I beams built-in at the ends in the x, z-plane are in process at the Istituto di Calcolo and will be published elsewhere in the near future.

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Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 0.1$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	31.40	28.06	24.30	19.84	14.03	0
0.9	33.80	30.44	26.53	21.79	15.30	0
0.8	35.75	32.26	28.16	23.15	16.25	0
0.7	37.53	33.82	29.48	24.22	17.04	0
0.6	39.35	35.37	30.76	25.21	17.80	0
0.5	41.40	37.12	32.20	26.33	18.65	0
0.4	43.82	39.21	33.97	27.74	19.70	0
0.3	46.69	41.78	36.20	29.57	21.04	0
0.2	50.07	44.86	38.95	31.89	22.74	0
0.1	53.95	48.50	42.28	34.77	24.86	0
0	58.30	52.65	46.16	38.21	27.44	0
- 0.1	63.02	57.25	50.54	42.18	30.51	0
- 0.2	67.99	62.18	55.33	46.63	34.09	0
- 0.3	73.03	67.27	60.38	51.45	38.16	0
- 0.4	77.92	72.31	65.50	56.51	42.71	0
- 0.5	82.39	77.04	70.46	61.63	47.71	0
- 0.6	86.13	81.17	74.99	66.60	53.10	0
- 0.7	88.79	84.33	78.76	71.16	58.82	0
- 0.8	89.97	86.15	81.41	75.04	64.78	0
- 0.9	89.23	86.16	82.53	77.90	70.89	0
- 1.0	86.07	83.90	81.67	79.39	77.03	0

Table 1

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 1$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	10.36	9.26	8.02	6.55	4.63	0
0.9	11.16	10.04	8.75	7.19	5.05	0
0.8	11.81	10.64	9.29	7.63	5.36	0
0.7	12.40	11.15	9.73	7.99	5.63	0
0.6	13.01	11.67	10.15	8.32	5.88	0
0.5	13.68	12.25	10.63	8.69	6.16	0
0.4	14.47	12.94	11.22	9.16	6.51	0
0.3	15.41	13.79	11.95	9.76	6.95	0
0.2	16.52	14.81	12.86	10.53	7.51	0
0.1	17.79	16.00	13.95	11.48	8.20	0
0	19.22	17.37	15.23	12.61	9.05	0
- 0.1	20.77	18.88	16.67	13.92	10.06	0
- 0.2	22.40	20.50	18.24	15.38	11.23	0
- 0.3	24.06	22.17	19.90	16.96	12.57	0
- 0.4	25.67	23.82	21.58	18.62	14.07	0
- 0.5	27.14	25.38	23.21	20.30	15.71	0
- 0.6	28.38	26.74	24.70	21.93	17.49	0
- 0.7	29.26	27.78	25.94	23.44	19.37	0
- 0.8	29.66	28.38	26.83	24.72	21.34	0
- 0.9	29.42	28.40	27.21	25.67	23.37	0
- 1.0	28.40	27.68	26.95	26.19	25.41	0

Table 2

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 2$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	7.66	6.85	5.93	4.84	3.42	0
0.9	8.25	7.42	6.47	5.30	3.72	0
0.8	8.73	7.86	6.86	5.64	3.95	0
0.7	9.16	8.24	7.18	5.90	4.15	0
0.6	9.61	8.62	7.50	6.14	4.34	0
0.5	10.11	9.05	7.85	6.42	4.55	0
0.4	10.70	9.56	8.28	6.77	4.81	0
0.3	11.39	10.19	8.83	7.21	5.14	0
0.2	12.21	10.94	9.50	7.78	5.54	0
0.1	13.14	11.82	10.30	8.48	6.07	0
0	14.19	12.82	11.24	9.31	6.69	0
- 0.1	15.33	13.93	12.30	10.27	7.43	0
- 0.2	16.52	15.12	13.45	11.34	8.29	0
- 0.3	17.74	16.35	14.67	12.50	9.27	0
- 0.4	18.92	17.57	15.91	13.72	10.37	0
- 0.5	20.00	18.71	17.11	14.96	11.58	0
- 0.6	20.91	19.71	18.21	16.16	12.89	0
- 0.7	21.56	20.48	19.13	17.27	14.28	0
- 0.8	21.86	20.93	19.78	18.23	15.73	0
- 0.9	21.71	20.96	20.08	18.94	17.24	0
- 1.0	20.98	20.44	19.90	19.34	18.76	0

Table 3

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 6$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	5.11	4.57	3.96	3.23	2.28	0
0.9	5.50	4.94	4.31	3.53	2.47	0
0.8	5.82	5.24	4.57	3.75	2.62	0
0.7	6.11	5.49	4.79	3.92	2.75	0
0.6	6.41	5.75	5.00	4.09	2.88	0
0.5	6.75	6.04	5.24	4.28	3.03	0
0.4	7.14	6.38	5.53	4.52	3.21	0
0.3	7.60	6.80	5.89	4.81	3.43	0
0.2	8.14	7.29	6.33	5.19	3.71	0
0.1	8.75	7.87	6.86	5.65	4.05	0
0	9.44	8.53	7.48	6.20	4.46	0
- 0.1	10.18	9.26	8.17	6.83	4.95	0
- 0.2	10.97	10.03	8.93	7.53	5.51	0
- 0.3	11.76	10.84	9.72	8.30	6.15	0
- 0.4	12.53	11.63	10.53	9.10	6.88	0
- 0.5	13.24	12.38	11.32	9.91	7.67	0
- 0.6	13.84	13.04	12.04	10.70	8.53	0
- 0.7	14.27	13.55	12.66	11.44	9.46	0
- 0.8	14.48	13.86	13.10	12.08	10.43	0
- 0.9	14.40	13.90	13.32	12.58	11.45	0
- 1.0	13.95	13.60	13.24	12.87	12.49	0

Table 4

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 10$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	4.43	3.96	3.43	2.80	1.98	0
0.9	4.76	4.28	3.73	3.05	2.14	0
0.8	5.03	4.53	3.95	3.24	2.26	0
0.7	5.28	4.75	4.14	3.40	2.38	0
0.6	5.55	4.98	4.34	3.54	2.50	0
0.5	5.84	5.23	4.54	3.71	2.63	0
0.4	6.18	5.53	4.79	3.92	2.79	0
0.3	6.58	5.89	5.11	4.17	2.98	0
0.2	7.04	6.31	5.49	4.50	3.22	0
0.1	7.57	6.81	5.94	4.89	3.51	0
0	8.16	7.37	6.47	5.36	3.86	0
- 0.1	8.80	7.99	7.06	5.90	4.27	0
- 0.2	9.46	8.65	7.70	6.49	4.76	0
- 0.3	10.14	9.33	8.38	7.14	5.30	0
- 0.4	10.80	10.01	9.07	7.83	5.92	0
- 0.5	11.40	10.65	9.74	8.52	6.60	0
- 0.6	11.91	11.21	10.36	9.20	7.34	0
- 0.7	12.29	11.66	10.89	9.84	8.14	0
- 0.8	12.48	11.94	11.28	10.40	8.99	0
- 0.9	12.43	11.99	11.49	10.85	9.88	0
- 1.0	12.07	11.76	11.45	11.13	10.80	0

Table 5

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 16$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	4.00	3.57	3.09	2.53	1.79	0
0.9	4.29	3.85	3.35	2.75	1.93	0
0.8	4.54	4.08	3.55	2.92	2.04	0
0.7	4.77	4.28	3.72	3.06	2.15	0
0.6	5.01	4.49	3.89	3.20	2.25	0
0.5	5.27	4.72	4.09	3.35	2.37	0
0.4	5.58	4.99	4.32	3.54	2.51	0
0.3	5.93	5.32	4.61	3.77	2.69	0
0.2	6.34	5.70	4.95	4.06	2.90	0
0.1	6.81	6.14	5.36	4.42	3.16	0
0	7.33	6.64	5.83	4.83	3.47	0
- 0.1	7.89	7.19	6.35	5.30	3.84	0
- 0.2	8.47	7.77	6.92	5.83	4.26	0
- 0.3	9.07	8.37	7.52	6.40	4.75	0
- 0.4	9.65	8.97	8.13	7.01	5.29	0
- 0.5	10.18	9.53	8.72	7.62	5.90	0
- 0.6	10.64	10.03	9.27	8.23	6.56	0
- 0.7	10.98	10.43	9.74	8.80	7.28	0
- 0.8	11.16	10.68	10.10	9.31	8.05	0
- 0.9	11.13	10.75	10.30	9.72	8.86	0
- 1.0	10.85	10.57	10.29	10.00	9.71	0

Table 6

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 24$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	3.73	3.34	2.89	2.36	1.67	0
0.9	4.00	3.59	3.13	2.56	1.79	0
0.8	4.23	3.80	3.31	2.71	1.90	0
0.7	4.44	3.99	3.48	2.84	1.99	0
0.6	4.67	4.18	3.64	2.97	2.10	0
0.5	4.92	4.40	3.82	3.12	2.21	0
0.4	5.21	4.65	4.04	3.30	2.35	0
0.3	5.54	4.95	4.30	3.52	2.51	0
0.2	5.92	5.31	4.62	3.79	2.71	0
0.1	6.35	5.71	4.99	4.12	2.95	0
0	6.83	6.17	5.42	4.50	3.24	0
- 0.1	7.34	6.67	5.90	4.94	3.58	0
- 0.2	7.88	7.20	6.42	5.42	3.97	0
- 0.3	8.41	7.75	6.97	5.94	4.41	0
- 0.4	8.94	8.29	7.52	6.50	4.91	0
- 0.5	9.42	8.81	8.07	7.06	5.47	0
- 0.6	9.83	9.27	8.58	7.62	6.08	0
- 0.7	10.13	9.64	9.02	8.15	6.75	0
- 0.8	10.30	9.89	9.36	8.63	7.46	0
- 0.9	10.27	9.97	9.56	9.03	8.22	0
- 1.0	10.01	9.84	9.58	9.31	9.03	0

Table 7

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 32$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	3.59	3.21	2.78	2.27	1.61	0
0.9	3.84	3.45	3.00	2.46	1.73	0
0.8	4.06	3.65	3.18	2.61	1.83	0
0.7	4.27	3.84	3.34	2.74	1.92	0
0.6	4.49	4.03	3.50	2.87	2.02	0
0.5	4.73	4.24	3.68	3.01	2.13	0
0.4	5.01	4.49	3.89	3.18	2.26	0
0.3	5.33	4.77	4.15	3.39	2.41	0
0.2	5.69	5.11	4.45	3.65	2.60	0
0.1	6.10	5.50	4.80	3.96	2.83	0
0	6.55	5.93	5.21	4.32	3.11	0
- 0.1	7.03	6.40	5.66	4.73	3.43	0
- 0.2	7.54	6.90	6.15	5.19	3.80	0
- 0.3	8.04	7.42	6.67	5.69	4.22	0
- 0.4	8.54	7.93	7.19	6.21	4.70	0
- 0.5	9.00	8.42	7.71	6.75	5.23	0
- 0.6	9.40	8.86	8.19	7.28	5.82	0
- 0.7	9.70	9.21	8.61	7.79	6.46	0
- 0.8	9.88	9.45	8.94	8.26	7.14	0
- 0.9	9.89	9.54	9.15	8.64	7.88	0
- 1.0	9.69	9.44	9.19	8.93	8.67	0

Table 8

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 40$

r \ p	0	0.2	0.4	0.6	0.8	1
1.0	3.51	3.14	2.72	2.22	1.57	0
0.9	3.75	3.37	2.93	2.40	1.68	0
0.8	3.97	3.57	3.10	2.54	1.78	0
0.7	4.17	3.75	3.26	2.67	1.87	0
0.6	4.38	3.93	3.41	2.80	1.97	0
0.5	4.62	4.14	3.59	2.94	2.08	0
0.4	4.89	4.38	3.80	3.11	2.21	0
0.3	5.20	4.66	4.05	3.32	2.36	0
0.2	5.55	4.98	4.34	3.57	2.55	0
0.1	5.95	5.35	4.69	3.87	2.78	0
0	6.38	5.77	5.08	4.22	3.04	0
- 0.1	6.84	6.22	5.52	4.62	3.35	0
- 0.2	7.33	6.71	5.99	5.06	3.70	0
- 0.3	7.82	7.20	6.48	5.53	4.11	0
- 0.4	8.30	7.70	6.99	6.03	4.57	0
- 0.5	8.74	8.17	7.48	6.55	5.08	0
- 0.6	9.12	8.59	7.94	7.06	5.65	0
- 0.7	9.42	8.94	8.35	7.56	6.27	0
- 0.8	9.59	9.18	8.68	8.01	6.94	0
- 0.9	9.61	9.27	8.88	8.39	7.66	0
- 1.0	9.43	9.18	8.94	8.69	8.44	0

Table 9

Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = 100$

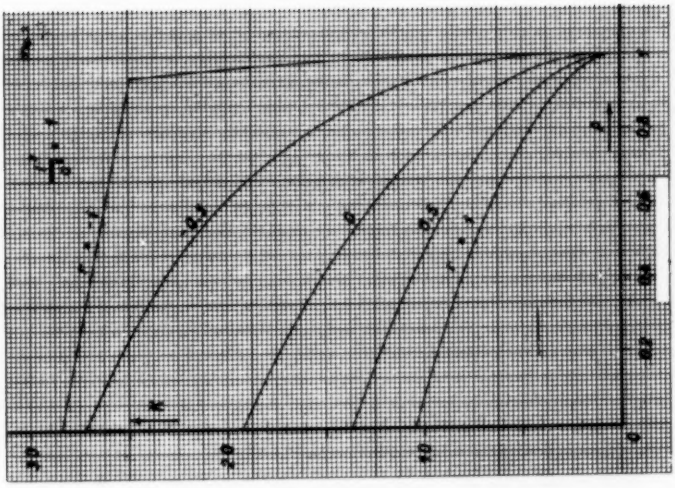
r \ p	0	0.2	0.4	0.6	0.8	1
1.0	3.29	2.95	2.55	2.08	1.47	0
0.9	3.50	3.16	2.74	2.24	1.56	0
0.8	3.70	3.34	2.89	2.37	1.65	0
0.7	3.89	3.51	3.04	2.49	1.74	0
0.6	4.10	3.68	3.19	2.61	1.84	0
0.5	4.33	3.88	3.36	2.75	1.95	0
0.4	4.59	4.10	3.56	2.91	2.08	0
0.3	4.87	4.36	3.79	3.11	2.22	0
0.2	5.20	4.66	4.06	3.34	2.40	0
0.1	5.55	5.00	4.38	3.61	2.60	0
0	5.94	5.37	4.73	3.93	2.84	0
- 0.1	6.35	5.77	5.12	4.29	3.12	0
- 0.2	6.77	6.20	5.54	4.68	3.43	0
- 0.3	7.20	6.64	5.98	5.11	3.80	0
- 0.4	7.61	7.08	6.43	5.56	4.21	0
- 0.5	8.00	7.50	6.87	6.02	4.67	0
- 0.6	8.34	7.88	7.29	6.49	5.18	0
- 0.7	8.61	8.19	7.66	6.94	5.75	0
- 0.8	8.78	8.42	7.97	7.36	6.38	0
- 0.9	8.83	8.53	8.18	7.74	7.06	0
- 1.0	8.72	8.49	8.27	8.04	7.81	0

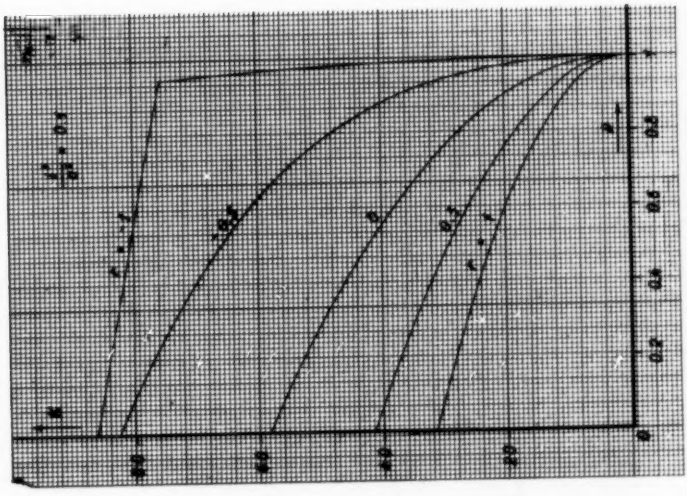
Table 10

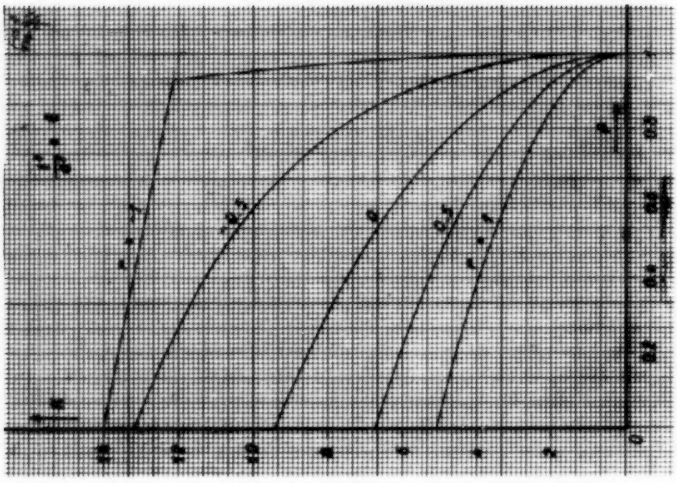
Values of $K = \frac{M_2 l}{\sqrt{BH}}$ versus $p = \frac{P l^2}{\pi^2 B}$ and $r = \frac{M_1}{M_2}$ for $\frac{l^2}{a^2} = \infty$

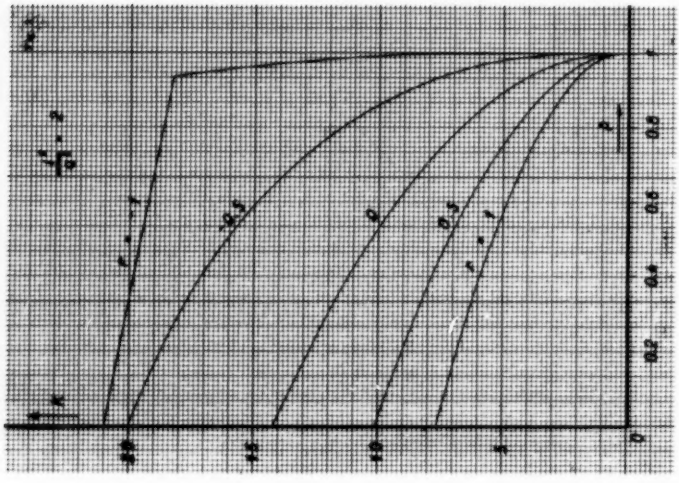
r \ p	0	0.2	0.4	0.6	0.8	1
1.0	3.14	2.81	2.43	1.99	1.40	0
0.9	3.31	2.99	2.60	2.13	1.49	0
0.8	3.48	3.16	2.75	2.25	1.58	0
0.7	3.68	3.33	2.89	2.37	1.66	0
0.6	3.89	3.51	3.04	2.49	1.76	0
0.5	4.12	3.70	3.21	2.63	1.86	0
0.4	4.37	3.92	3.40	2.79	1.98	0
0.3	4.64	4.16	3.62	2.97	2.12	0
0.2	4.93	4.44	3.88	3.19	2.28	0
0.1	5.24	4.75	4.16	3.44	2.48	0
0	5.56	5.08	4.48	3.73	2.70	0
- 0.1	5.90	5.44	4.83	4.05	2.96	0
- 0.2	6.25	5.81	5.20	4.40	3.25	0
- 0.3	6.61	6.19	5.58	4.78	3.57	0
- 0.4	6.97	6.56	5.97	5.18	3.94	0
- 0.5	7.32	6.92	6.36	5.59	4.34	0
- 0.6	7.66	7.25	6.73	6.00	4.78	0
- 0.7	7.96	7.52	7.06	6.41	5.26	0
- 0.8	8.19	7.73	7.34	6.80	5.77	0
- 0.9	8.25	7.85	7.54	7.14	6.31	0
- 1.0	8.03	7.85	7.65	7.44	6.89	0

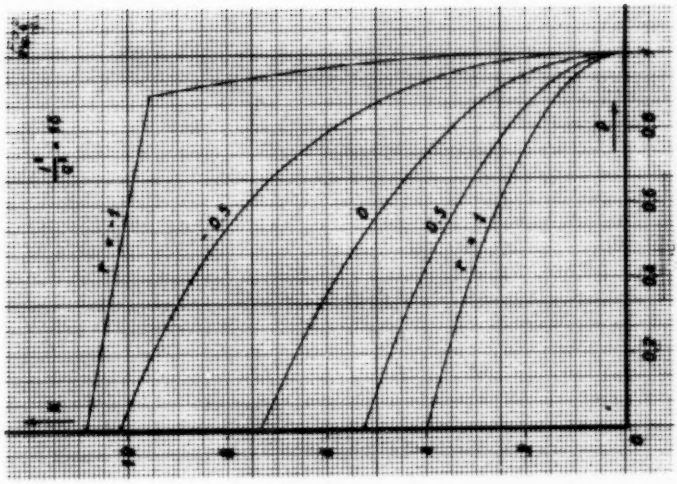
Table 11

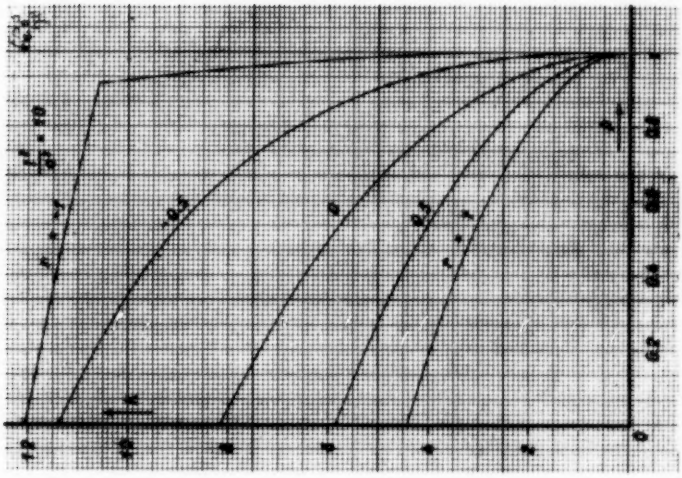


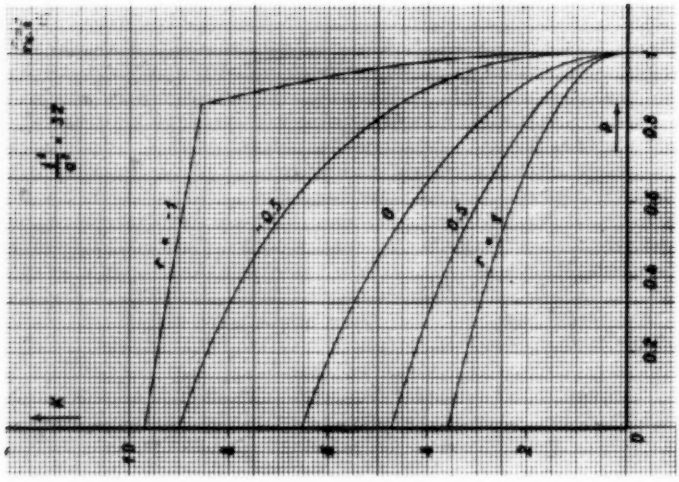








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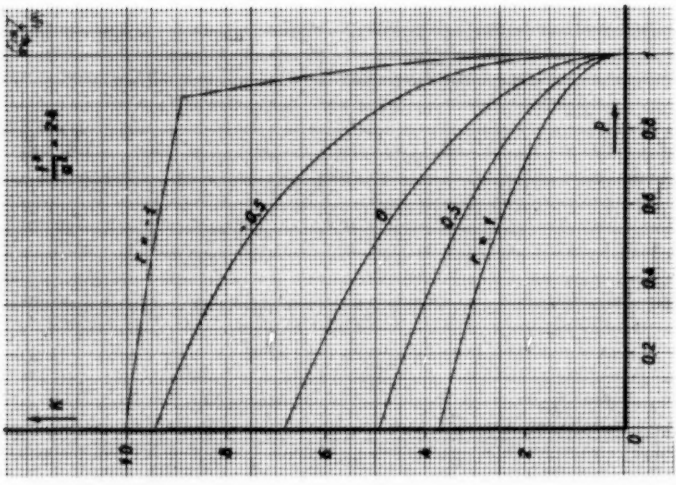
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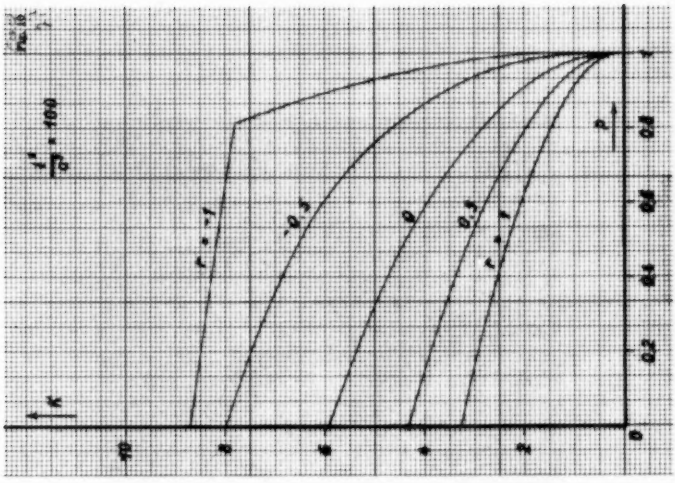

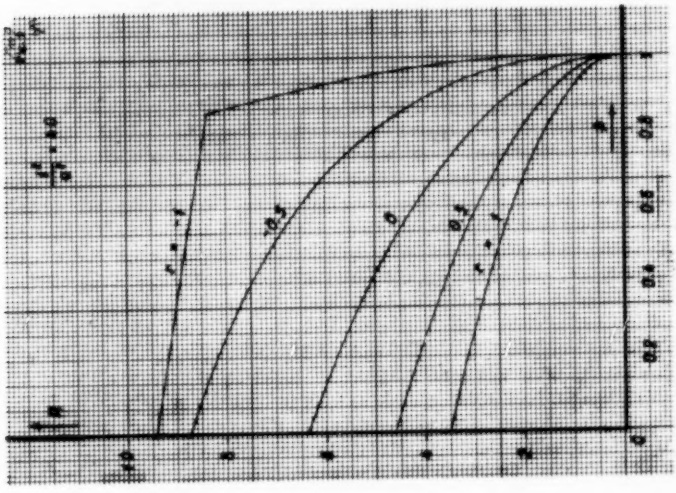

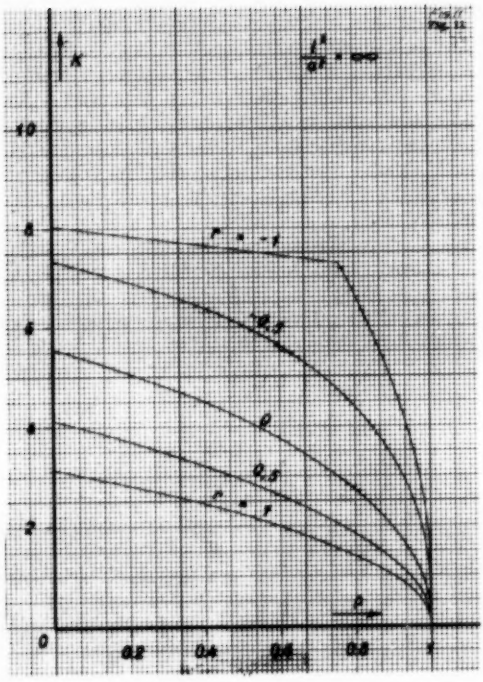
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0.6
0.8
1
r = -1
-0.5
0
0.5
r = 1

Fig 12

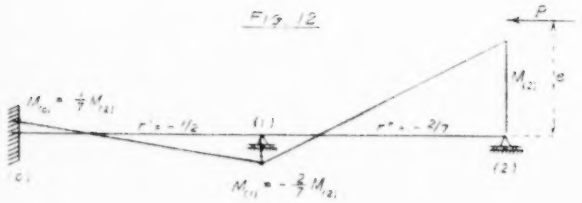
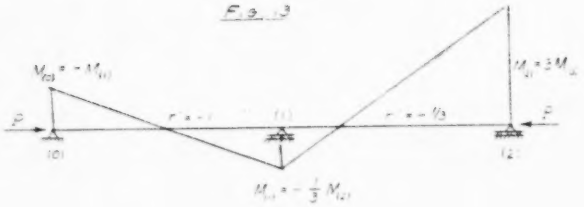


Fig 13



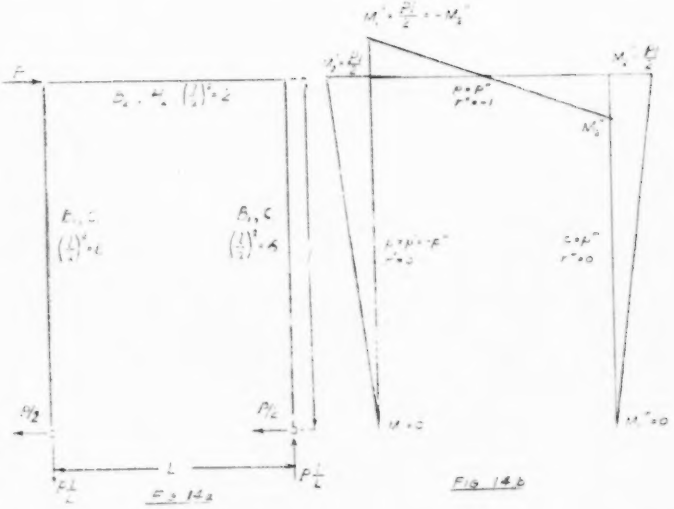


Fig 14a

Fig 14b

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